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# Inverse Boundary Problems in Two Dimensions

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*Dedicated to Prof. Hans Triebel on the occasion of his 65th birthday*

**Abstract.** In this paper we survey some of the recent progress on inverse boundary problems in two dimensions. The common theme is the use of inverse scattering for a $\overline{\partial}\partial$ type system in two dimensions.

## 1. Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary (many of the results we will describe are valid for domains with Lipschitz boundaries). The electrical conductivity of $\Omega$ is represented by a bounded and positive function $\gamma(x)$. In the absence of sinks or sources of current the equation for the potential is given by

$$\operatorname{div}(\gamma \nabla u) = 0 \text{ in } \Omega \tag{1.1}$$

since, by Ohm's law, $\gamma \nabla u$ represents the current flux.

Given a potential $f \in H^{\frac{1}{2}}(\partial\Omega)$ on the boundary the induced potential $u \in H^1(\Omega)$ solves the Dirichlet problem

$$\begin{aligned} \operatorname{div}(\gamma \nabla u) &= 0 \text{ in } \Omega, \\ u\big|_{\partial\Omega} &= f. \end{aligned} \tag{1.2}$$

The Dirichlet to Neumann (DN) map, or voltage to current map, is given by

$$\Lambda_\gamma(f) = \left(\gamma \frac{\partial u}{\partial \nu}\right)\Big|_{\partial\Omega} \tag{1.3}$$

where $\nu$ denotes the unit outer normal to $\partial\Omega$.

The inverse problem is to determine $\gamma$ knowing $\Lambda_\gamma$. This problem is known also as *Electrical Impedance Tomography*. It arose originally in geophysical prospection [ZK94]. More recently it has been proposed as a valuable diagnostic tool in medicine [CIN99]. The mathematical formulation and the first mathematical results are due to Calderón [Cal80]. The survey paper [Uhl99] summarizes many of the developments in inverse boundary boundary problems up to 1997 which were pioneered by Calderón's contribution. In this paper we will discuss the progress in the last few years for the two-dimensional problem based on the $\overline{\partial}\partial$ method which was originally developed by Beals and Coifman [BC88] in order to solve, via inverse scattering, the nonlinear Davey-Stewartson system.

The main breakthrough in EIT in the two-dimensional case is due to A. Nachman who proved in [Nac96] that one can uniquely determine conductivities in $W^{2,p}(\Omega)$ for some $p > 1$. Moreover he proposed a method to reconstruct the conductivity from the DN map. Significant progress has been made in developing a numerical algorithm based on Nachman's reconstruction method [SMI00].

We describe an alternative approach to the identifiability result of Nachman developed in [BU97], approach that allows for less regular conductivities. Section 2 outlines this approach and an extension to complex conductivities. In Section 3 we make a further analysis of this method for the case of $C^{1+\epsilon}(\overline{\Omega})$ conductivities which is needed for Section 4. In Section 4 we discuss stability estimates proven in [BBR01] and a reconstruction method proposed in [KT01], both of which consider $C^{1+\epsilon}(\overline{\Omega})$ conductivities. In Section 5 we outline the use of the $\overline{\partial}\partial$ method in studying the DN map (or more generally the set of Cauchy data) for the Pauli Hamiltonian [KU02] and in Section 6 for any first order perturbation of the Laplacian with no zeroth order terms [CY98]. Finally in Section 7 we mention some open problems.

## 2. The $\overline{\partial}\partial$ system

In this section we describe an extension of Nachman's result to $W^{1,p}(\Omega)$ conductivities with $p > 2$, result due to Brown and the author [BU97]. We follow an earlier approach of Beals and Coifman [BC88] and L. Sung [Sung94a,b,c] who studied scattering for a first order system whose principal part is $\begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix}$.

The main result of [BU97] is:

**Theorem 2.1.** *Let $n = 2$. Let $\gamma \in W^{1,p}(\Omega)$, $p > 2$, $\gamma$ strictly positive. Assume $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Then $\gamma_1 = \gamma_2$ in $\Omega$.*

As mentioned earlier, the proof of Theorem 2.1 first reduces the conductivity equation to a first order system, which we will call the $\overline{\partial}\partial$ system. We define

$$q = -\frac{1}{2}\partial \log \gamma \tag{2.1}$$

and a matrix potential $Q$ by

$$Q = \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix}. \tag{2.2}$$

Let also $D$ be the operator

$$D = \begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix}, \tag{2.3}$$

where $\overline{\partial} = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2}), \partial = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$.

An easy calculation shows that, if $u$ satisfies the conductivity equation $\operatorname{div}(\gamma\nabla u) = 0$, then

$$\begin{pmatrix} v \\ w \end{pmatrix} = \gamma^{\frac{1}{2}} \begin{pmatrix} \partial u \\ \overline{\partial} u \end{pmatrix} \tag{2.4}$$

solves the system

$$D \begin{pmatrix} v \\ w \end{pmatrix} - Q \begin{pmatrix} v \\ w \end{pmatrix} = 0. \tag{2.5}$$

In [BU97] Brown and Uhlmann construct matrix solutions of (2.5) of the form

$$\psi(z,k) = m(z,k) \begin{pmatrix} e^{izk} & 0 \\ 0 & e^{-i\overline{z}k} \end{pmatrix}, \tag{2.6}$$

where $z = x_1 + ix_2$, $k \in \mathbb{C}$ with $m \to 1$ as $|z| \to \infty$ in a sense to be described below. A simple calculation shows that $m$ from (2.6) satisfies in $\Omega$ the following equation

$$D_k m - Qm = 0, \tag{2.7}$$

where $D_k$ is the operator

$$D_k = \begin{pmatrix} (\overline{\partial} - i\overline{k}) & 0 \\ 0 & (\partial + ik) \end{pmatrix}.$$

In order to explain the construction of $m$ we need a few more definitions. Let

$$\Lambda_k(z) = \begin{pmatrix} e(z,k) & 0 \\ 0 & e(z,-\overline{k}) \end{pmatrix}, \quad e(z,k) = e^{i(zk+\overline{z}\overline{k})}$$

and for any matrix $A$, define the following operator

$$E_k A = E_k \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & e(z,-\overline{k})a_{12} \\ a_{21}e(z,k) & a_{22} \end{pmatrix}.$$

Notice that

$$D_k = E_k^{-1} D E_k. \tag{2.8}$$

Let $D^{-1}$ be the operator

$$D^{-1} = \begin{pmatrix} \overline{\partial}^{-1} & 0 \\ 0 & \partial^{-1} \end{pmatrix},$$

where

$$\overline{\partial}^{-1} f(z) = \frac{1}{2\pi i} \int \frac{f(w)}{z-w} dw \wedge \overline{w}$$

and

$$\partial^{-1} f(z) = \frac{1}{2\pi i} \int \frac{f(w)}{\overline{z}-\overline{w}} dw \wedge \overline{w}.$$

We have from (2.8) that $D_k^{-1} = E_k^{-1} D^{-1} E_k$. We look for solutions of (2.7) among the solutions of the integral equation

$$(I - D_k^{-1} Q) m(z) = I, \tag{2.9}$$

where $I$ is the $2 \times 2$ identity matrix. For a $2 \times 2$ matrix $A$, let $A^d$ and $A^{off}$ denote its diagonal respectively off-diagonal part. If

$$J = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

we define the operator $\mathcal{J}$ by

$$\mathcal{J}A = [J, A] = 2JA^{off} = -2A^{off}J. \tag{2.10}$$

To end with the preliminary notation, we recall the definition of the weighted $L^p$ space

$$L^p_\alpha(\mathbb{R}^2) = \{f \ : \ \int (1+|x|^2)^\alpha |f(x)|^p dx < \infty\}.$$

The next result gives the solvability of (2.7) in an appropriate space.

**Theorem 2.2.** *Let $Q \in L^p(\mathbb{R}^2)$, $p > 2$, and compactly supported. Assume that $Q$ is a hermitian matrix. Choose $r$ so that $\frac{1}{p} + \frac{1}{r} > \frac{1}{2}$ and then $\beta$ so that $\beta r > 2$. Then the operator $(I - D_k^{-1}Q)$ is invertible in $L^r_{-\beta}$. Moreover the inverse is differentiable in $k$ in the strong operator topology.*

Theorem 2.2 implies the existence of solutions of the form (2.6) with $m - 1 \in L^r_{-\beta}(\mathbb{R}^2)$.

We remark that the proof of 2.2 consists in showing that the integral equation (2.9) is of Fredholm type in $L^r_{-\beta}$. The fact that it has been a trivial kernel follows by showing that if $(I - D_k^{-1}Q)n(z,k) = 0$, then $n \in L^p$, for all $p > 2$, satisfies a pseudoanalytic equation in the $z$-variable. By the standard Liouville's theorem for pseudoanalytic equations with coefficients in $L^p$, $p > 2$, it follows that $n = 0$.

Next we compute $\frac{\partial}{\partial \overline{k}} m(z,k)$.

**Theorem 2.3.** *Let $m$ be the solution of (2.7) with $m - 1 \in L^r_{-\beta}(\mathbb{R}^2)$. Then*

$$\frac{\partial}{\partial \overline{k}} m(z,k) - m(z,\overline{k})\Lambda_k(z) S_Q(k) = 0 \tag{2.11}$$

*where the scattering data $S_Q$ is given by* ([BC88])

$$S_Q(k) = -\frac{1}{\pi}\mathcal{J}\int_{\mathbb{R}^2} E_k Q m \, d\mu, \tag{2.12}$$

*where $d\mu$ denotes Lebesgue measure in $\mathbb{R}^2$.*

A further calculation shows that

$$S_Q(k) = \frac{i}{\pi}\int_{\mathbb{R}^2} \begin{pmatrix} 0 & e(z,-\overline{k})q(z)m_{22}(z,k) \\ -e(z,k)\overline{q}(z)m_{11}(z,k) & 0 \end{pmatrix}. \tag{2.13}$$

The behavior of $m$ in the $k$-variable is given by the following result:

**Theorem 2.4.** *Let $Q \in L^p(\mathbb{R}^2)$, $p > 2$, and compactly supported. Then there exists $R = R(Q)$ so that for all $q > \frac{2p}{p-2}$,*

$$\sup_z \|m(z,\cdot) - 1\|_{L^q\{k;|k|>R\}} \leq C\|Q\|^2_{L^p}$$

*where the constants depend on $p, q$ and the diameter of the support of $Q$.*

*Outline of proof of Theorem 2.1* We know [Ale88, KV84, Nac88, SyU88] that if $\gamma_i \in W^{1,p}(\Omega)$ and $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\partial^\alpha \gamma_1\Big|_{\partial\Omega} = \partial^\alpha \gamma_2\Big|_{\partial\Omega}$ $\forall\ |\alpha| \leq 1$. Therefore we can extend $\gamma_i \in W^{1,p}(\mathbb{R}^2)$, $\gamma_1 = \gamma_2$ in $\mathbb{R}^2 \setminus \Omega$ and $\gamma_i = 1$ outside a large ball. Thus $Q_i \in L^p(\mathbb{R}^2)$, $i = 1, 2$. The proof follows the following steps.

**Step 1.** $\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow S_{Q_1} = S_{Q_2} := S$. With these extensions, we observe that for each $j$ the scattering data $S_{Q_j}(k)$, $j = 1, 2$, has the representation

$$S_{Q_j}(k) = -2J \int_{\mathbf{R}^2} \begin{pmatrix} 0 & \bar\partial\psi_j^{12} e^{-iz\bar k} \\ \partial\psi_j^{21} e^{i\bar z k} & 0 \end{pmatrix} d\mu(z)$$

$$= -2J \left[ \int_{\mathbf{R}^2\setminus\Omega} \begin{pmatrix} 0 & \bar\partial\psi_j^{12} e^{-iz\bar k} \\ \partial\psi_j^{21} e^{i\bar z k} & 0 \end{pmatrix} d\mu(z) + \int_{\partial\Omega} \begin{pmatrix} 0 & \bar\nu\psi_j^{12} e^{-iz\bar k} \\ \nu\psi_j^{21} e^{i\bar z k} & 0 \end{pmatrix} d\mu(z) \right].$$

The formula for $S_{Q_j}$ uses the complexified normal to the boundary

$$\nu(z) = \nu_1(z) + i\nu_2(z), \quad \overline{\nu}(z) = \nu_1(z) - i\nu_2(z) \tag{2.14}$$

with $(\nu_1(z), \nu_2(z))$ the unit outer normal at $z \in \partial\Omega$.

From this expression for $S_{Q_j}$, $j = 1, 2$, we see that if we can show

$$\psi_1(z,k) = \psi_2(z,k) \quad \text{in } \mathbf{R}^2\backslash\bar\Omega, \tag{2.15}$$

then $S_{Q_1} = S_{Q_2}$.

The last formula follows by using a very similar argument to Lemma 2.6 in [SyU86].

**Step 2.** Let $\widetilde m = m_1 - m_2$. Using the $\overline\partial$-equation (2.11) and Step 1 we conclude that

$$\frac{\partial}{\partial\overline k}\widetilde m(z,k) - \widetilde m(z,\overline k)\Lambda_k(z)S(k) = 0\ . \tag{2.16}$$

With the elements of $\widetilde m$ we form the following four functions

$$\begin{aligned} u_\pm(k,z)) &= \widetilde m_{11}(z,k) \pm \overline{\widetilde m_{12}(z,\overline k)} \\ v_\pm(k,z)) &= \widetilde m_{21}(z,k) \pm \overline{\widetilde m_{22}(z,\overline k)} \end{aligned}$$

each of which lies in $L^q(\mathbb{R}^2)$ in the $k$-variable and satisfies, for a fixed $z$, a pseudoanalytic equation in the $k$-variable,

$$\frac{\partial}{\partial\overline k} w(z,k) = r(z,k)\overline{w(z,k)} \tag{2.17}$$

where $r(z,k)$ is some component of $S$ multiplied by a complex coefficient of norm 1.

**Step 3.** In [BU97] it was shown that, for $Q \in L^p_c$ for some $p > 2$ with $Q^* = Q$, we have that

$$\int trS_Q S_Q^* \leq \int trQQ^*.$$

This shows that $S_Q \in L^2$. Consequently, for each fixed $z$ we have the map $k \to r(z,k)$ is in $L^2(\mathbb{R}^2)$.

**Step 4.** Prove that $u_\pm = v_\pm = 0$, hence $\widetilde{m} = 0$ or $m_1 = m_2$. Then it is easy to show $Q_1 = Q_2$ and therefore $\gamma_1 = \gamma_2$.

To do this we need the following generalization of the Liouville Theorem for pseudoanalytic functions proven in [BU97].

**Lemma 2.5.** *Let $f \in L^2(\mathbb{R}^2)$ and $w \in L^p(\mathbb{R}^2)$ for some finite $p$. Assume that $we^{\overline{\partial}^{-1}f}$ is analytic. Then $w = 0$.*

Let us define

$$\begin{aligned} \widetilde{u}_\pm &= u_\pm e^{\overline{\partial}^{-1} r} \\ \widetilde{v}_\pm &= v_\pm e^{\overline{\partial}^{-1} r}. \end{aligned} \tag{2.18}$$

It is easy to check that $\widetilde{u}_\pm$ and $\widetilde{v}_\pm$ are analytic. By the lemma above we conclude that $u_\pm = v_\pm = 0$ which in turn gives $m_1 = m_2$. It is easy to show $Q_1 = Q_2$ and therefore $\gamma_1 = \gamma_2$, concluding the proof of Theorem 2.1.

The idea of the proof of Lemma 2.5 is the observation that since $r \in L^2(\mathbb{R}^2)$, $f = \overline{\partial}^{-1} r$ is in $VMO(\mathbb{R}^2)$ (the space of functions with vanishing mean oscillation) and thus is $O(\log |z|)$ as $|z| \to \infty$. Hence $e^f w \in L^{\widetilde{p}}$ for $\widetilde{p} > p$. By Liouville's theorem it follows that $e^f w = 0$. The details can be found in [BU97].

Theorem 2.1 was extended to complex conductivities with small imaginary part in [Fra00], using the $\overline{\partial}\partial$ method. Complex conductivities with small imaginary part arise naturally when considering Maxwell's equations for time harmonic waves with small frequency.

## 3. $C^{1+\epsilon}$ conductivities

The difficulty in obtaining stability estimates or a reconstruction procedure for conductivities in $W^{1,p}$, $p > 2$, is to prove stability estimates for $w(z,k)$ as in (2.17), in terms of the scattering data $S_Q(k) \in L^2$ or to reconstruct $w(z,k)$ in terms of $S_Q(k)$. All of the other steps are constructive and stability estimates can be obtained. If one assumes that the conductivity is more regular then $S_Q$ is in a better space than $L^2$. This allows to carry out a constructive proof and give stability estimates. We will assume that the potential $Q$ is in $C^\epsilon(\mathbf{R}^2)$ for some $\epsilon > 0$ and derive the stronger decay properties of the scattering matrix than being in $L^2$. More precisely we have the following result proven in [BBR01]:

**Lemma 3.1.** *Let $Q \in C^\epsilon(\mathbb{R}^2)$ be compactly supported. Then $S_Q \in L^r(\mathbb{R}^2)$ for any $r > 4/(2+\epsilon)$.*

The proof of the Lemma is based on some unpublished results of Brown and the author [BU96] which we outline below. We construct more explicitly the solutions of the integral equation (2.9). In particular we show that if the potential $Q$ is in $C^\epsilon(\mathbf{R}^2)$ and is compactly supported, then the special solutions are in $C^\epsilon(\mathbf{R}^2)$ with respect to $z$, uniformly in $k$. Next we observe that the diagonal part of $m$ is $C^{1+\epsilon}(\mathbf{R}^2)$. This regularity is essential in proving the Lemma.

We will use the following norm for the space $C^\alpha(\mathbf{R}^2)$, $0 < \alpha < 1$, $\|f\|_{C^\alpha} = \|f\|_{L^\infty} + |f|_{C^\alpha}$ where we use $|f|_{C^\alpha}$ for the semi-norm

$$|f|_{C^\alpha} = \sup_{z \neq w} \frac{|u(z) - u(w)|}{|z - w|^\alpha}$$

and then $C^{1+\alpha}(\mathbf{R}^2)$ is the collection of functions for which the norm $\|u\|_{L^\infty(\mathbf{R}^2)} + \|\nabla u\|_{C^\alpha(\mathbf{R}^2)}$ is finite.

The strategy is to show that for large $k$, the special solutions are given by a convergent Neumann series. For small $k$, we appeal to the Fredholm theory as in the proof of Theorem 2.2. This method does not give quantitative estimates without a more thorough analysis. We remark that one feature of the operator $D_k^{-1}$ is that the diagonal part of $D_k^{-1}F$ is smoother than the off-diagonal part. This is because the off-diagonal part is multiplied by an exponential. This elementary observation will be used several times.

We begin with some elementary but useful estimates for the action of $D_k^{-1}Q$ on the spaces $C^\epsilon(\mathbf{R}^2)$. There estimates rely on the well-known fact that when $\epsilon = 1 - 2/p$ (and then $p > 2$)

$$|f|_{C^\epsilon} \leq C\|f\|_{L^p}. \tag{3.1}$$

**Proposition 3.2.** *Let $F \in C^\epsilon(\mathbf{R}^2)$, and suppose $F$ is supported in a ball of radius $R$. Then for $0 < \epsilon < 1/2$,*

$$\|D_k^{-1}F\|_{C^\epsilon(\mathbf{R}^2)} \leq C(R, \epsilon)\|F\|_{C^\epsilon}. \tag{3.2}$$

*Also,*

$$\|D_k^{-1}F^{off}\|_{L^\infty} \leq C(1 + |k|)^{-\epsilon}\|F^{off}\|_{C^\epsilon} \tag{3.3}$$

*and*

$$\|D_k^{-1}F^d\|_{C^\epsilon} \leq C\|F\|_{L^\infty}. \tag{3.4}$$

*All constants depend on $R$ and $\epsilon$.*

*Proof.* The third estimate (3.4) is a consequence of the sharper estimate (3.1). Thanks to (3.4), we only need to prove (3.2) for $F = F^d$. The second estimate (3.3) will be established as a byproduct of our proof of (3.2).

When $|k| \leq 1$, the estimates (3.2) and (3.3) are easy. Thus we suppose $F = F^{off}$ and $|k| \geq 1$. We first note that

$$\|E_k F\|_{C^\epsilon} \leq C(1 + |k|)^\epsilon \|F\|_{C^\epsilon}. \tag{3.5}$$

Next, we choose a smooth function $\varphi$ supported in the unit ball, centered at 0, and with $\int \varphi = 1$. As usual, we set $\varphi_t(z) = t^{-2}\varphi(z/t)$. We let $0 < t \leq 1$ and split

$F = F_s + F_r$ where the smooth part is $F_s = \varphi_t * F$ and the rough part $F_r = F - F_s$. We observe that $\|F_r\|_{L^\infty} \le C|F|_{C^\epsilon} t^\epsilon$, and for $|\alpha| \ge 1$, $\|\frac{\partial^\alpha}{\partial x^\alpha} F_s\|_{L^\infty} \le C|F|_{C^\epsilon} t^{\epsilon - |\alpha|}$. Since $0 < t \le 1$, we have that $F_s$ and $F_r$ supported in a ball of radius $R+1$. We consider $D^{-1}E_k F_r$ and observe that since $F_r$ is compactly supported, we may use (3.5) to conclude

$$\|D^{-1}E_k F_r\|_{C^\epsilon} \;\le\; C(R,\epsilon)\|E_k F_r\|_{L^\infty} \;\le\; Ct^\epsilon.$$

For $D^{-1}E_k F_r$, we observe that $D\Lambda_k^{-1}(z) = ik \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \Lambda_k^{-1}(z)$ and then integrating by parts gives

$$\begin{aligned} D^{-1}E_k F_s(z) &= \frac{1}{\pi}\int \begin{pmatrix} z-w & 0 \\ 0 & \bar{z}-\bar{w} \end{pmatrix}^{-1} \begin{pmatrix} -ik & 0 \\ 0 & ik \end{pmatrix}^{-1} D\Lambda_k^{-1}(w)F_s(w)d\mu(w) \\ &= \begin{pmatrix} ik & 0 \\ 0 & -ik \end{pmatrix}^{-1} \Lambda_k^{-1}(z)F_s(z) \\ &\quad - \frac{1}{\pi}\int \begin{pmatrix} z-w & 0 \\ 0 & \bar{z}-\bar{w} \end{pmatrix}^{-1} \begin{pmatrix} -ik & 0 \\ 0 & ik \end{pmatrix}^{-1} \Lambda_k^{-1}(w)DF_s(w)d\mu(w) \\ &= I + II. \end{aligned}$$

By (3.5) and the fact that $\|F_s\|_{C^\epsilon} \le \|F\|_{C^\epsilon}$, we have that

$$\|I\|_{C^\epsilon} \le \frac{C}{|k|}(1+|k|)^\epsilon \|F\|_{C^\epsilon}.$$

Using the support restrictions on $F_s$, we also have that

$$\|II\|_{C^\epsilon} \;\le\; \frac{C}{|k|}\|\Lambda_k DF_s\|_{L^\infty} \;\le\; \frac{C}{|k|} t^{\epsilon-1}\|F\|_{C^\epsilon}.$$

Combining these observations gives

$$\|D^{-1}E_k F\|_{C^\epsilon} \le C\|F\|_{C^\epsilon}\left(t^\epsilon + \frac{(1+|k|)^\epsilon}{|k|} + \frac{t^{\epsilon-1}}{|k|}\right).$$

Choosing $t = 1/|k|$ gives

$$\|D^{-1}E_k F\|_{C^\epsilon} \le C(1+|k|)^{-\epsilon}\|F\|_{C^\epsilon}$$

for $0 < \epsilon \le 1/2$. The estimate (3.3) follows immediately. The estimate (3.2) follows from (3.5). □

One reason it is convenient to study the map $f \to D_k^{-1}Qf$ on the spaces $C^\epsilon$ is that these spaces form an algebra under pointwise multiplication. The following elementary inequality

$$\|QF\|_{C^\epsilon} \le C\|F\|_{C^\epsilon}\|Q\|_{C^\epsilon},\ 0 < \epsilon < 1$$

will be used several times below.

**Proposition 3.3.** *Suppose $Q$ is $C^\epsilon$, compactly supported and $Q^d = 0$. Then there is a constant $C_0$ so that if*

$$C_0(1+|k|)^{-2}\|Q\|^2_{C^\epsilon} = \theta < 1, \qquad \text{then} \qquad m(\cdot,k) = \sum_{j=0}^{\infty}(D_k^{-1}Q)^j(I)$$

*and the series converges in $C^\epsilon(\mathbf{R}^2)$ and $\|m(\cdot,k)\|_{C^\epsilon} \le \frac{C}{1-\theta}$.*

*Proof.* We will show that

$$\|(D_k^{-1}Q)^{2j}(1)\|_{C^\epsilon} \le C(1+|k|)^{-\epsilon}\|Q\|^{2j}_{C^\epsilon}. \tag{3.6}$$

We note that if $F = F^d$, then

$$(D_k^{-1}Q)^2F = D^{-1}QD_k^{-1}QF.$$

Now by (3.4), we have that

$$\|D_k^{-1}QF\|_{L^\infty} \le C(1+|k|)^{-\epsilon}\|Q\|_{C^\epsilon}\|F\|_{C^\epsilon}.$$

Since $Q$ is compactly supported, we have

$$\|D^{-1}QF\|_{C^\epsilon} \le C(R)\|F\|_{L^\infty}\|Q\|_{L^\infty}.$$

Combining these estimates gives

$$\|(D_k^{-1}Q)^2F\|_{C^\epsilon} \le C(1+|k|)^{-\epsilon}\|F\|_{C^\epsilon}\|Q\|^2_{C^\epsilon}.$$

Our claim (3.6) and hence the proposition follow from this estimate and (3.3). □

**Corollary 3.4.** *Suppose $Q$ is in $C^\epsilon$, compactly supported and $Q^d = 0$. Then there is a constant $C$ so that*

$$\|m(\cdot,k)\|_{C^\epsilon} \le C.$$

*Here, $C$ depends on $Q$.*

*Proof.* For $k$ large, this follows from the previous proposition. To see that we also have a bound for small $k$, we observe that since $Q$ is compactly supported, $k \to (I - D_k^{-1}Q)^{-1}(I)$ is a continuous map into $L^r_{-\beta}$ and thus on each compact subset $K \subset \mathbf{C}$,

$$\sup_{k\in K}\|m(\cdot,k)\|_{L^r_{-\beta}} < \infty. \tag{3.7}$$

Using that $Q$ is bounded and compactly supported and the equation

$$m = 1 + D_k^{-1}Qm$$

we may use (3.1) to improve (3.7) and obtain

$$\sup_{k\in K}\|m(\cdot,k)\|_{C^\epsilon} < \infty.$$

Combining this with our result for large $k$ gives the Corollary. □

Our next step is to observe that the diagonal part of $m$ is smoother. This is an easy consequence of the integral equation.

**Proposition 3.5.** *If $Q \in C^\epsilon(\mathbb{R}^2)$ and is compactly supported, then $m^d$ is in $C^{1+\epsilon}(\mathbb{R}^2)$ and we have*

$$\|m^d(\cdot,k)\|_{C^{1+\epsilon}} \leq C.$$

*Proof.* We have that $m - D_k^{-1}Qm \equiv 1$ and that $m \in C^\epsilon(\mathbf{R}^2)$. In particular,

$$m^d = \begin{pmatrix} \bar\partial^{-1}q_{12}m_{21} & 0 \\ 0 & \partial^{-1}q_{21}m_{12} \end{pmatrix}.$$

We have $\bar\partial^{-1}, \partial^{-1} : C^\epsilon(\mathbf{R}^2) \to C^{1+\epsilon}(\mathbf{R}^2)$. Thus the entries of $m^d$ are $C^{1+\epsilon}(\mathbf{R}^2)$ since $Qm$ is $C^\epsilon(\mathbf{R}^2)$. We obtain the $L^\infty$-estimates for $m$ since $Qm$ is compactly supported. □

**Corollary 3.6.** *Let $Q$ be in $C^\epsilon(\mathbb{R}^2)$ and compactly supported. Then the special solutions constructed in Theorem 2.2 and Proposition 3.3 satisfy*

$$\|m(\cdot,k) - 1\|_{L^\infty} \leq C|k|^{-\epsilon}.$$

*Proof.* We have $m(\cdot,k) \in C^\epsilon(\mathbf{R}^2)$ from Corollary 3.4. We observe that

$$m^{off} = D_k^{-1}Qm^d,$$

then the estimate (3.3) implies $\|m^{off}\|_{L^\infty} \leq C|k|^{-\epsilon}$. Now we have

$$m^d = 1 + D^{-1}Qm^{off}$$

and since $Q$ is compactly supported, we have

$$\|m^d - 1\|_{L^\infty} \ \leq\ C(Q)\|m^{off}\|_{L^\infty} \ \leq\ C|k|^{-\epsilon}.$$

□

Using these estimates and $L^2$ estimates for pseudodifferential operators with non-smooth symbols proven in [CM78] one can easily conclude the proof of Lemma 3.1. For more details see [BBR01].

## 4. Stability estimates and reconstruction

Conditional stability estimate for Nachman's approach for conductivities with one derivatives was proved in [Liu97], while stability estimate using the $\overline{\partial}\partial$ approach for conductivities in $C^{1+\epsilon}$, $\epsilon > 0$, was shown in [BBR01]. It is the latter result that is stated below.

**Theorem 4.1.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary and assume that $\gamma_1$ and $\gamma_2$ are two conductivities in $\overline{\Omega}$ such that for $i = 1, 2$,*

(i) *There exists a constant $C > 0$ such that for every $x \in \Omega$*

$$\frac{1}{C} < \gamma_i < C, \tag{4.1}$$

(ii) *$\gamma_i \in C^{1+\epsilon}(\overline{\Omega})$ for some $\epsilon > 0$ and there exists $M > 0$ such that*

$$\|\gamma_i\|_{C^{1+\epsilon}} \leq M. \tag{4.2}$$

*If* $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}}$ *is small enough, then*

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \le C\omega\left(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}}\right), \tag{4.3}$$

*where* $\|\cdot\|_{\frac{1}{2},-\frac{1}{2}}$ *denotes the operator norm as operators from* $H^{\frac{1}{2}}(\partial\Omega)$ *to* $H^{-\frac{1}{2}}(\partial\Omega)$ *and* $\omega : [0,\delta] \to \mathbb{R}$ *is such that*

$$\omega(t) \le |\log t|^{-\alpha}, \tag{4.4}$$

*for some* $\alpha > 0$.

For details on the proof we refer to [BBR01].

Next we focus our attention on the reconstruction method. In [Nac96] a reconstruction method is proposed for conductivities in $W^{2,p}(\Omega)$, $p > 2$. In the remaining of the section we outline the reconstruction method using the $\overline{\partial}\partial$ approach developed in [KT01].

With $Q \in C^\epsilon(\overline{\Omega})$, the natural Cauchy data for the system (2.5) is

$$\mathcal{C}_Q = \{(v|_{\partial\Omega}, w|_{\partial\Omega}) : (v,w) \in C^{1+\epsilon}(\overline{\Omega}) \times C^{1+\epsilon}(\overline{\Omega}),\ (D-Q)(v,w)^T = 0\}.$$

We remark that the method in [KT01] also shows that $Q$ is uniquely determined by $\mathcal{C}_Q$. However, it is still unclear how to reconstruct it. In the case when $Q$ comes from a conductivity (2.2) $\mathcal{C}_Q$ is determined by $\Lambda_\gamma$.

**Theorem 4.2.** *Let* $\Omega \subset \mathbb{R}^2$ *be bounded with smooth boundary and the conductivity* $\gamma \in C^{1+\epsilon}(\overline{\Omega})$ *be such that* $0 < c \le \gamma$ *for some constant* $c$. *Then* $\gamma$ *can be reconstructed from* $\Lambda_\gamma$

The reconstruction method is done in three steps.

**Step 1.** Determine $\psi$ on $\partial\Omega$. There are two ideas involved. The first consists of a complete characterization of $\mathcal{C}_Q$ in terms of $\Lambda_\gamma$ as follows. Let $\dot{C}^{1+\epsilon}(\overline{\Omega})$ be the space of functions in $C^{1+\epsilon}(\overline{\Omega})$ which integrate to zero along the boundary. Let $\partial_s^{-1}$ denote an inverse to the tangential field $\partial_s$ along the boundary. Notice that it is well defined when acting on boundary maps which integrate to zero. The complexified normal $\nu$ was defined in (2.14).

**Theorem 4.3.** *Let* $Q \in C^\epsilon(\overline{\Omega})$ *compactly supported in* $\Omega$ *be as in (2.5). Then*

$$\begin{aligned}\mathcal{C}_Q = \{(h_1,h_2) \in \dot{C}^{1+\epsilon}(\partial\Omega) \times \dot{C}^{1+\epsilon}(\partial\Omega) :\\ i\Lambda_\gamma\partial_s^{-1}(\nu h_1 - \overline{\nu} h_2) = (\nu h_1 + \overline{\nu} h_2)\}.\end{aligned} \tag{4.5}$$

The proof relies on the equivalence between the boundary value problem for the conductivity equation (1.2) with $f = i\partial_s^{-1}(\nu h_1 - \overline{\nu} h_2)$ and the boundary value problem for the equation (2.5) with the boundary data $(u,v)|_{\partial\Omega} = (h_1,h_2)$ and on the following straightforward relation on the boundary $\partial\Omega$,

$$\begin{pmatrix} v \\ w \end{pmatrix}\Bigg|_{\partial\Omega} = \frac{1}{2}\begin{pmatrix} \overline{\nu} & -i\overline{\nu} \\ \nu & i\nu \end{pmatrix}\begin{pmatrix} \Lambda_\gamma(f) \\ \partial_s(f) \end{pmatrix}. \tag{4.6}$$

The second idea takes into account the behaviour of the complex geometrical optics $\psi$ for $z$ near infinity. Notice that the first row has entries which are analytic outside $\Omega$, while the second row has entries which are anti-analytic outside $\Omega$.

Due to the symmetries

$$m_{11}(z,k) = \overline{m_{22}(z,\overline{k})}, \qquad m_{21}(z,k) = \overline{m_{12}(z,\overline{k})}, \tag{4.7}$$

which follows from the differential equations, the asymptotic for the columns of $m$ and the uniqueness in Theorem 2.2, it suffices to reconstruct the first column $(\psi_{11}, \psi_{21})^T$ of $\psi(z,k)$.

For every $(\zeta, z) \in \mathbb{C}$ with $\zeta \neq z$ we introduce $g_k(\zeta, z) = \frac{1}{\pi}\frac{e^{-ik(\zeta-z)}}{\zeta-z}$, a Green kernel for $\overline{\partial}$ which also takes into account exponential growth at infinity. Using $g_k(\cdot, z)$ with $z \in \partial\Omega$, we define the single layer potentials $\mathcal{S}_k$ and $\overline{\mathcal{S}}_k$ as boundary integral operators by

$$\mathcal{S}_k f(z) = \int_{\partial\Omega} f(\zeta) g_k(\zeta, z) d\zeta, \ \ \overline{\mathcal{S}}_k f(z) = \int_{\partial\Omega} f(\zeta) \overline{g}_k(\zeta, z) d\zeta, \tag{4.8}$$

where the integrals are understood in the sense of principal value. It is a classical result in singular integral theory that these operators are well defined bounded operators from $C^{1+\epsilon}(\partial\Omega)$ to $C^{1+\epsilon}(\partial\Omega)$, see [Musk53].

The following result completely characterizes the traces on $\partial\Omega$.

**Theorem 4.4.** *The only pair* $(h_1, h_2) \in \dot{C}^{1+\epsilon}(\partial\Omega) \times \dot{C}^{1+\epsilon}(\partial\Omega)$ *which satisfies*

$$\begin{pmatrix} I - i\mathcal{S}_k & 0 \\ 0 & I - i\overline{\mathcal{S}}_k \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 2e^{izk} \\ 0 \end{pmatrix} \tag{4.9}$$

*together with*

$$(I - i\Lambda_\gamma \partial_s^{-1})(\nu h_1)(z) = (I + i\Lambda_\gamma \partial_s^{-1})(\overline{\nu} h_2)(z), \tag{4.10}$$

*is* $(\psi_{11}(\cdot, k), \psi_{21}(\cdot, k))|_{\partial\Omega}$.

The idea of the proof is based on the uniqueness of direct scattering. The pair $(h_1, h_2)$ may be extended inside $\Omega$ since they belong to the Cauchy data $\mathcal{C}_Q$, see Theorem 4.3. Moreover, this extension is unique due to the ellipticity of the system. Outside $\Omega$ they can be explicitly extended by

$$v(z) = -\frac{1}{2i}\int_{\partial\Omega} h_1(\zeta) g_k(\zeta, z) d\zeta + e^{izk}, \tag{4.11}$$

respectively

$$w(z) = -\frac{1}{2i}\int_{\partial\Omega} h_2(\zeta) \overline{g}_k(\zeta, z) d\zeta. \tag{4.12}$$

It is easy to see that they are analytic respectively anti-analytic and that they satisfy the decay condition from Theorem 2.2. The system (2.5) is also satisfied across the boundary $\partial\Omega$, since Plemelj's formula [Musk53] ensures continuity. The analyticity, respectively anti-analyticity is obtained using Morera's theorem.

**Step 3.** Using the formula (2.13) we find the scattering matrix $S_Q$.

Next we solve the $\overline{\partial}$-equation (2.11). In [BU97] the fact that $S_Q \in L^2(\mathbb{R}^2)$ suffices. Here we will use the further decay on $S_Q$ given by Lemma 3.1.

To simplify notations we introduce the operator $(\partial/\partial\overline{k})^{-1} = \partial_{\overline{k}}^{-1}$ defined by

$$\partial_{\overline{k}}^{-1} f(k) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(k')}{k-k'}dk'.$$

We can now write the integral equations for $m$ and prove unique solvability of these. We denote below by $S_{ij}$ the components of the matrix $S_Q$.

**Lemma 4.5.** *Let $Q \in C^\epsilon(\mathbb{R}^2)$ be compactly supported, and let $z \in \mathbb{C}$ be fixed. Then for $q > 4/\epsilon$ the equation*

$$(I - \partial_{\overline{k}}^{-1}(e(z,-k)S_{21}\cdot))(m - I) = \partial_{\overline{k}}^{-1}(e(z,-k)S_{21}), \tag{4.13}$$

*has a unique solution $m(z,\cdot)$ such that $m(z,\cdot) - I \in L^q(\mathbb{R}^2)$ given by*

$$m(z,k) = I + (I - \partial_{\overline{k}}^{-1}(e(z,-k)S_{21}\cdot))^{-1}\partial_{\overline{k}}^{-1}(e(z,-k)S_{21}). \tag{4.14}$$

*Moreover, $m(z,\cdot) - I \in C^\alpha(\mathbb{R}^2)$, for $\alpha < (1+\epsilon)/2$.*

*Proof.* Since $S_{21} \in L^2(\mathbb{R}^2)$ and $|e(z,-k)| = 1$ we know (see [Nac96]) that $\partial_{\overline{k}}^{-1}(e(z,-k)S_{21}\cdot)$ is a compact operator in $L^s(\mathbb{R}^2)$ for $2 < s < \infty$. Furthermore, by Lemma 3.1, since $S_{21} \in L^r(\mathbb{R}^2)$ for some $4/(2+\epsilon) < r < 2$, it follows by using the Hardy-Littlewood-Sobolev inequality that $\partial_{\overline{k}}^{-1}(e(z,-k)S_{21}) \in L^q(\mathbb{R}^2)$ for $q > \epsilon/2$. The unique solvability of (4.13) in $L^q(\mathbb{R}^2)$ follows from the Fredholm alternative. The fact that $(I + \partial_{\overline{k}}^{-1}(e(z,-k)S_{21}\cdot))$ has trivial kernel in $L^q(\mathbb{R}^2)$ is a consequence of Liouville's theorem for pseudoanalytic functions with coefficients in $L^r(\mathbb{R}^2) \cap L^{r'}(\mathbb{R}^2)$ (see [Vek62]).

To prove the Hölder continuity of $m(z,\cdot) - I$ we use the fact, that convolution by $1/z$ maps $L^p(\mathbb{R}^2)$ into $C^\alpha(\mathbb{R}^2)$ for $1 < p < \infty$, $\alpha = 1 - 1/p_0$ and $\max(2,p) < p_0 < \infty$, see [SuU93]. □

**Step 3.** Recover $\gamma$ in $\Omega$. If we solve (2.11) for $m(z,k)$ for each fixed $z \in \mathbb{C}$, then $Q$ can be found by

$$Q(z) = \lim_{k_0\to\infty} \mu(B_r(0))^{-1} \int_{\{k:|k-k_0|<r\}} D_k m(z,k)d\mu(k).$$

This would solve the inverse problem for (2.5).

However, as observed in [BBR01] we can reconstruct $\gamma$ directly. First solve (2.11) with $S_Q(k)$ replaced by $S_Q(-\overline{k})^T$ and call the solution $\tilde{m}(z,k)$. Then define

$$\tilde{m}_+(z,k) = \tilde{m}_{11}(z,k) + \overline{\tilde{m}_{12}(z,\overline{k})}.$$

The conductivity $\gamma$ is given by the formula [KT01]

$$\gamma^{1/2}(z) = \mathrm{Re}(\tilde{m}_+(z,0)). \tag{4.15}$$

## 5. The Pauli Hamiltonian

The Pauli Hamiltonian describes particles in a magnetic field with spin. In two dimensions it is a direct sum of the pair of operators

$$H_{\vec{A},q}u := \sum_{j=1}^{2}\left(\frac{1}{i}\frac{\partial}{\partial x_j} - A_j\right)^2 u \pm Bu - qu \tag{5.1}$$

where $\vec{A}$ denotes the magnetic potential, $B = \operatorname{rot}\vec{A}$ is the magnetic field, and $q$ is the electrical potential (see for instance Chapter 6 of [CFKS87]). Thus both direct and inverse problems for the Pauli Hamiltonian consider separately both signs in $B$ in (5.1). For simplicity we will choose the plus sign. All of the results below are also valid for the minus sign in $B$ with minor changes in the arguments.

We first describe the inverse boundary problem we consider. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. We consider the Pauli Hamiltonian given by a real-valued vector field $\vec{A} = (A_1, A_2) \in W^{1,p}(\Omega)$ and an electric potential $q \in L^p(\Omega)$, $p > 2$,

$$H_{\vec{A},q}u := \sum_{j=1}^{2}\left(\frac{1}{i}\frac{\partial}{\partial x_j} - A_j\right)^2 u + Bu - qu = 0 \quad \text{in } \Omega \tag{5.2}$$

where $B = \operatorname{rot}\vec{A}$. Fix $\alpha = \frac{p-2}{p}$ throughout this section. The set of Cauchy data of the solutions of (5.2) is given by

$$\begin{aligned}\mathcal{C}_{\vec{A},q} := \{(f,g) \in C^{1,\alpha}(\partial\Omega) \times C^{\alpha}(\partial\Omega) :\ &\text{there exists } u \in C^{1,\alpha}(\overline{\Omega}) \\ \text{such that } H_{\vec{A},q}u = 0,\ u|_{\partial\Omega} = f,\ &((\nabla - i\vec{A})u)|_{\partial\Omega}\cdot\nu = g\}.\end{aligned} \tag{5.3}$$

Here $\nu$ denotes the unit normal to $\partial\Omega$. In the case that 0 is not a Dirichlet eigenvalue for $H_{\vec{A},q}$, $\mathcal{C}_{\vec{A},q}$ is the graph of the Dirichlet-to-Neumann (DN) map

$$\Lambda_{\vec{A},q} : C^{1,\alpha}(\partial\Omega) \to C^{\alpha}(\partial\Omega). \tag{5.4}$$

The inverse boundary value problem we consider in this section is whether we can determine $\vec{A}$ and $q$ from $\mathcal{C}_{\vec{A},q}$.

It was observed in [Sun93a, Sun93b] that there is a gauge invariance in the problem. That is, if $\varphi \in C^1(\overline{\Omega})$ with $\varphi|_{\partial\Omega} = 1$, $\nabla\varphi|_{\partial\Omega} = 0$, then

$$\mathcal{C}_{\vec{A}+\nabla\varphi,q} = \mathcal{C}_{\vec{A},q}.$$

Therefore we can recover at best the magnetic field, $\operatorname{rot}\vec{A}$, and $q$ from the DN map.

In [KU02] it was proven the following semiglobal identifiability result:

**Theorem 5.1.** *Let $\vec{A}_j \in W_0^{1,p}(\Omega)$, $j = 1,2$, $\operatorname{rot}\vec{A}_1 \in W_0^{1,p}(\Omega)$, $q_1 \in W^{1,p}(\Omega)$, $q_2 \in L^p(\Omega)$, $p > 2$. For each $M > 0$, there exists $\epsilon(M,\Omega,p) > 0$ such that if $\|\vec{A}_1\|_{L^p(\Omega)} \le M$ and $\|q_1\|_{W^{1,p}(\Omega)} \le \epsilon$ and*

$$\mathcal{C}_{\vec{A}_1,q_1} = \mathcal{C}_{\vec{A}_2,q_2}, \tag{5.5}$$

*we conclude*

$$\operatorname{rot}\vec{A}_1 = \operatorname{rot}\vec{A}_2 \quad \textit{and} \quad q_1 = q_2 \quad \textit{in } \Omega. \tag{5.6}$$

$W_0^{1,p}(\Omega)$ denotes the space of $W^{1,p}(\Omega)$-functions whose boundary traces are zero.

Observe that no smallness condition is assumed on the electric potential $q_2$. We also remark that the only place where we need that the magnetic potential has boundary trace zero is in the proof of Lemma 5.6. If we assume further regularity in the magnetic and electrical potentials then Theorem D of [NSU] (which is also valid in two dimensions) allows to extend the magnetic and electrical potentials to $\mathbb{R}^2$ with compact support. More precisely we have:

**Theorem 5.2.** *Let $\vec{A}_j$, $q_j \in C^\infty(\overline{\Omega})$, $j = 1, 2$, and $p > 2$. There exists $\epsilon(\Omega, p) > 0$ such that if $\|q_1\|_{W^{1,p}(\Omega)} \le \epsilon$ and*

$$\mathcal{C}_{\vec{A}_1, q_1} = \mathcal{C}_{\vec{A}_2, q_2}, \tag{5.7}$$

*we conclude*

$$\operatorname{rot}\vec{A}_1 = \operatorname{rot}\vec{A}_2 \quad \textit{and} \quad q_1 = q_2 \quad \textit{in } \Omega. \tag{5.8}$$

In [Sun93b] Sun proved in two dimensions for the Schrödinger equation in a magnetic field that if $\|\operatorname{rot}\vec{A}_j\|_{W^{1,\infty}(\Omega)}$ $(j = 1, 2)$ is small enough and $q_j$ $(j = 1, 2)$ are in an open and dense set in an appropriate topology, then we can determine uniquely $\operatorname{rot}\vec{A}_j$ and $q_j$ from the DN map associated to the magnetic potentials and electrical potentials.

We remark that in dimensions $n \ge 3$ a global identifiability result of the magnetic field and electrical potential was proven in [NSU95] for the Schrödinger equation in a magnetic field assuming some smoothness conditions on the coefficients. We also note that even if we assume that $\vec{A}_1 = 0$ in Theorem 5.1, it is unknown whether we can recover $q$ from $\mathcal{C}_q := \mathcal{C}_{\vec{0}, q}$, the set of Cauchy data for the standard Schrödinger equation, although several partial results have been proven ([BU97], [Kan00], [Nac96], [SuU91], [SuU93], [SyU86]).

A particular case of Theorem 5.1 is when the electrical potential in (5.1) is zero. Thus we obtain the following global uniqueness result:

**Corollary 5.3.** *Let $\vec{A}_j \in W_0^{1,p}(\Omega)$, $j = 1, 2$, $\operatorname{rot}\vec{A}_1 \in W_0^{1,p}(\Omega)$, $q_1 = 0$, $q_2 \in L^p(\Omega)$, $p > 2$. For each $M > 0$, there exists $\epsilon(M, \Omega, p) > 0$ such that if $\|\vec{A}_1\|_{L^p(\Omega)} \le M$*

$$\mathcal{C}_{\vec{A}_1, 0} = \mathcal{C}_{\vec{A}_2, q_2}, \tag{5.9}$$

*we conclude*

$$\operatorname{rot} A_1 = \operatorname{rot} A_2 \quad \textit{and} \quad q_1 = q_2 = 0 \quad \textit{in } \Omega. \tag{5.10}$$

As a consequence of Theorem 5.2, a similar result to Corollary 5.3 holds with $\vec{A}_j \in C^\infty(\overline{\Omega})$, $j = 1, 2$, without the assumption that the magnetic potentials have zero boundary trace.

In the case that the magnetic potentials are zero, we conclude the following semiglobal identifiability result:

**Corollary 5.4.** *Let $q_1 \in W^{1,p}(\Omega)$, $q_2 \in L^p(\Omega)$, $p > 2$. There exists $\epsilon(\Omega, p) > 0$ such that if $\|q_1\|_{W^{1,p}(\Omega)} \le \epsilon$ and $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$, then $q_1 = q_2$.*

Corollary 5.4 was known previously if $q_1$ and $q_2$ are both a priori close to constant [SyU86]. As far as we know, even the case $q_1 = 0$ was previously unknown.

The method of proof of Theorem 5.1 is by reducing the problem to a similar one for a second order equation which can be factored in terms of $\overline{\partial}$ and $\partial$.

We rewrite (5.2) in the form

$$(\overline{\partial} + \bar{a})(\partial - a)u - \tilde{q}u = 0 \quad \text{in } \Omega \tag{5.11}$$

where

$$a := \frac{1}{2}(A_2 + iA_1), \quad \tilde{q} = \frac{1}{4}q. \tag{5.12}$$

We define the set of Cauchy data associated to (5.11) by

$$\begin{aligned}\mathcal{C}_{a,\tilde{q}} := \{(f, g) \in C^{1,\alpha}(\partial\Omega) \times C^{\alpha}(\partial\Omega) : u|_{\partial\Omega} = f,\\ ((\partial - a)u)|_{\partial\Omega} = g,\ u \in C^{1,\alpha}(\overline{\Omega}) \text{ a solution of (5.11) }\}.\end{aligned} \tag{5.13}$$

Theorem 5.1 is then a consequence of

**Theorem 5.5.** *Let $a_j \in W^{1,p}(\Omega)$, $\tilde{q}_1 \in W^{1,p}(\Omega)$, $\tilde{q}_2 \in L^p(\Omega)$, $p > 2$, $j = 1, 2$. For each $M > 0$, there exists $\epsilon(M, \Omega, p) > 0$ such that if $\|a_1\|_{L^p(\Omega)} \le M$ and $\|\tilde{q}_1\|_{W^{1,p}(\Omega)} \le \epsilon$ and*

$$\mathcal{C}_{a_1,\tilde{q}_1} = \mathcal{C}_{a_2,\tilde{q}_2}, \tag{5.14}$$

*then*

$$\tilde{q}_1 = \tilde{q}_2 \quad \text{and} \quad \overline{\partial}^{-1}\bar{a}_1 + \partial^{-1}a_1 = \overline{\partial}^{-1}\bar{a}_2 + \partial^{-1}a_2 \quad \text{in } \Omega. \tag{5.15}$$

Of course (5.15) implies that

$$\operatorname{rot} a_1 = \frac{1}{2}(\overline{\partial}\bar{a}_1 + \partial a_1) = \operatorname{rot} a_2 = \frac{1}{2}(\overline{\partial}\bar{a}_2 + \partial a_2) \quad \text{in } \Omega. \tag{5.16}$$

*Sketch of proof of Theorem 5.5*

The method of proof of Theorem 5.5 reduces (5.11) to a first order system and follows the outline of Section 2.

Consider the equation

$$(\overline{\partial} + \bar{a})(\partial - a)u - qu = 0 \quad \text{in } \Omega\ . \tag{5.17}$$

Let $u$ be a solution of (5.17) and set $w := (\partial - a)u$. Then the equation (5.17) takes the form

$$\left[\begin{pmatrix} \overline{\partial} + \bar{a} & 0 \\ 0 & \partial - a \end{pmatrix} - \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}\right]\begin{pmatrix} w \\ u \end{pmatrix} = 0. \tag{5.18}$$

By conjugating the equation (5.18), we have

$$\begin{pmatrix} e^{-\overline{\partial}^{-1}\bar{a}} & 0 \\ 0 & e^{\partial^{-1}a} \end{pmatrix}\left[\begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix} - \begin{pmatrix} 0 & e^{Ta}q \\ e^{-Ta} & 0 \end{pmatrix}\right]\begin{pmatrix} e^{\overline{\partial}^{-1}\bar{a}} & 0 \\ 0 & e^{-\partial^{-1}a} \end{pmatrix}\begin{pmatrix} w \\ u \end{pmatrix} = 0 \tag{5.19}$$

where

$$Ta(z) := \overline{\partial}^{-1}\bar{a} + \partial^{-1}a.$$

Set

$$D = \begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix}$$

and

$$\widetilde{Q} = \begin{pmatrix} 0 & e^{Ta}q \\ e^{-Ta} & 0 \end{pmatrix}.$$

We are seeking special solutions of the system

$$(D - \widetilde{Q})\psi = 0 \quad \text{in } \Omega \tag{5.20}$$

in the form

$$\psi = \begin{pmatrix} e^{-\overline{\partial}^{-1}\bar{a}} & 0 \\ 0 & e^{\partial^{-1}a} \end{pmatrix} m(z,k) \begin{pmatrix} e^{izk} & 0 \\ 0 & e^{-i\bar{z}k} \end{pmatrix} \tag{5.21}$$

where $m(z,k)$ is a $2 \times 2$ matrix-valued function in $\Omega$.

We will not repeat all the arguments of the proof that follows pretty much the steps of Section 2. We mention, though, that in order to prove that the boundary value of the solutions $\psi_j$, $j = 1, 2$ in Step 1 one needs the following Lemma due to Sun (equation (3.44) in [Sun93b]).

**Lemma 5.6.** *If $\mathcal{C}_{a_1,q_1} = \mathcal{C}_{a_2,q_2}$, then $\partial^{-1}a_1 = \partial^{-1}a_2$ on $\partial\Omega$.*

Z. Sun proved this lemma under the assumption that the DN maps are the same. However, exactly the same argument works with the assumption $\mathcal{C}_{a_1,q_1} = \mathcal{C}_{a_2,q_2}$. Also Sun's proof works under the weaker regularity assumptions assumed in this section on the electrical and magnetic potentials.

## 6. Determination of convection terms

In this section we sketch a proof, using the method of Sections 2 and 4, of a Theorem of Cheng and Yamamoto [CY98] (for an announcement see [CY00]) regarding the unique identification of the convection term $B = (B_1, B_2) \in L^p(\Omega), p > 2$ entering

$$\Delta u + B \cdot \nabla u = 0 \quad \text{in } \Omega \tag{6.1}$$

in terms of the Dirichlet-to-Neumann map $\Lambda_B$ defined by

$$\Lambda_B(u) = \frac{\partial u}{\partial \nu} \tag{6.2}$$

with $u$ solution to (6.1).

We define

$$b := \frac{1}{4}(B_1 + iB_2).$$

Then, (6.1) becomes

$$\overline{\partial}\partial u + \bar{b}\partial u + b\overline{\partial} u = 0 \quad \text{in } \Omega.$$

Put $v := \partial u$ and $w := \overline{\partial} u$. We obtain

$$\left[\begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix} + \begin{pmatrix} \bar{b} & b \\ \bar{b} & b \end{pmatrix}\right]\begin{pmatrix} v \\ w \end{pmatrix} = 0. \tag{6.3}$$

Conjugating (6.3) we have

$$\begin{pmatrix} e^{-\overline{\partial}^{-1}\bar{b}} & 0 \\ 0 & e^{-\partial^{-1}b} \end{pmatrix}\left[\begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix} - \begin{pmatrix} 0 & e^{Tb}b \\ e^{\overline{Tb}}\bar{b} & 0 \end{pmatrix}\right]\begin{pmatrix} e^{\overline{\partial}^{-1}\bar{b}} & 0 \\ 0 & e^{\partial^{-1}b} \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} = 0$$

where

$$Tb(z) := \overline{\partial}^{-1}\bar{b} - \partial^{-1}b.$$

Put $q := e^{Tb}b$. Then we have the following system of equations:

$$\left[\begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix} - \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}\right]\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \quad \text{in } \Omega. \tag{6.4}$$

By Theorem 2.2, for each $k \in \mathbb{C}$, there exists a solution of (6.4) of the form

$$\psi(z,k) = m(z,k)\begin{pmatrix} e^{izk} & 0 \\ 0 & e^{-i\bar{z}k} \end{pmatrix}.$$

If $B^{(1)}$ and $B^{(2)}$ are two convection coefficients and satisfy $\Lambda_{B^{(1)}} = \Lambda_{B^{(2)}}$, then by using arguments similar to Lemma 5.6, $\overline{\partial}^{-1}B^{(1)} = \overline{\partial}^{-1}B^{(2)}$ on $\partial\Omega$. As in Step 1 of the proof of Theorem 2.1 outlined in Section 2 one gets that the scattering matrix $S_{Q_j}(k)$ corresponding to (6.4) coincide. It then follows from the Liouville Theorem for pseudo-analytic functions, Theorem 3.1 of [BU97], that

$$m^{(1)}(z,k) = m^{(2)}(z,k), \quad k \in \mathbb{C}.$$

Therefore, we have

$$e^{\overline{\partial}^{-1}\bar{b}^{(1)} - \partial^{-1}b^{(1)}}b^{(1)} = e^{\overline{\partial}^{-1}\bar{b}^{(2)} - \partial^{-1}b^{(2)}}b^{(2)}$$

and hence

$$b^{(1)} = b^{(2)}$$

concluding the proof of the Theorem.

## 7. Open problems

In this section we mention some open problems directly related to the results of this paper.

- **Bounded measurable conductivities**
  The Dirichlet-to-Neumann map is well defined for $L^\infty$ conductivities. Is it possible to extend Theorem 2.1, Theorem 4.1 and Theorem 4.2 to this case? A less ambitious project would be to extend these results to conductivities $\gamma \in W^{1,2}(\Omega)$. If this is possible then it is very likely that one can show that the inequality in Step 3 of Section 2 is an equality providing a Plancherel formula for the map $Q \longrightarrow S_Q$. This was done by Brown [B01] for the case

that $\|Q\|_{L^2}$ is small enough. It is known [BC88] that there is a Plancherel identity for Schwartz potentials.

- **Numerical algorithm**
  Develop a numerical algorithm based on the reconstruction method proposed in [KT01].
- **Complex conductivities**
  Show that Theorem 2.1, Theorem 4.1 and Theorem 4.2 extend to complex conductivities, without the assumption of smallness in the imaginary part.
- **The Schrödinger equation**
  Extend Theorem 5.1 (or Theorem 5.2) to a global uniqueness result, without the smallness assumption on the potential.
- **Other elliptic systems**
  Can the $\bar{\partial}\partial$ method used in this paper be applied to other elliptic systems? One important case to consider is the isotropic elasticity system in two dimensions. Up to the present time only a local uniqueness result is known [NU93].

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